**Chapter 03: Complex Differentiation and Cauchy-Riemann Equations**

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## 3.1 Derivatives

If is single-valued in some region of the plane, the **derivate** of is given as:

Of course, this depends on the limit existing. If this is true, then is said to be **differentiable**.

## 3.2 Analytic Functions

If a derivative of exists at all points within the region , then it is said to be **analytic** in . The terms **regular** and **holomorphic** may also be used.

A function is said to be analytic at a point if there exists a neighbourhood, , in which a derivative exists at all points.

## 3.3 Cauchy-Riemann Equations ([Video](https://www.youtube.com/watch?v=GvOzQXIbVts))

Consider we have a function . If we break this function into its real and imaginary parts,

Here, the real part, , can be denoted as a function of , as in . Similarly, the imaginary part, , can be denoted as a function of , as in . Thus, we can write .

For a function, , to be analytic with a region , and must satisfy the **Cauchy-Riemann Equations** (CR equations) in the region.

Using the equations we derived above,

If the partial derivates are continuous in , then these equations are sufficient conditions for to be analytic in . They are useful since, if we are given , we can find from the Cauchy-Riemann equations and vice versa. For example, if we have , we can find and . From these, we can find and . After this, we simply integrate as .

Any functions that satisfy the Cauchy-Riemann equations are called **Conjugate Functions** (CF). Since and both do this, they are Conjugate Functions to each other.

## 3.4 Harmonic Functions ([Video](https://www.youtube.com/watch?v=a31_8bdNWVs))

For any analytic function, , we just proved that the Cauchy Riemann equations are satisfied. Now, we want to take the **second order partial derivates** of and with respect to and and prove that they give us .

[from the Cauchy Reimann equations]

Similarly, we can also prove that in the same manner.

Thus, we have proven two things:

Remember that and are both from the same equation, . Let us denote this equation as . If we do this, then the two equations above can be combined to:

This equation is called **Laplace’s equation**. It can also be written as where . is called the **Laplacian**. Functions that satisfy Laplace’s equation in a region are called **Harmonic Functions** (HF) in the region. Thus, for all analytic functions, the functions that define their real and imaginary parts, i.e. and , are harmonic functions. Since and are also Conjugate Functions, they can be called **Harmonic Conjugate Functions** (HCF)

## 3.7 Rules for Differentiation

Suppose , and are analytic functions of . For these equations:

1. , where is any constant.
2. , given that .
3. If and ,

Similarly, if , and ,

The above rule is called the **chain rule** for differentiation of complex functions.

1. If has a single-valued inverse, , then and
2. If and ,

Similar rules can also be found for differentials. E.g.

## 3.10 L’Hospital’s Rule

Let and be **analytic** in a region containing the point and let , but . In this case,

If , this rule can be further extended.

It is said that the left-hand side has the **intermediate form** of . Other limits that have such intermediate forms, such as , , , , and can be found by modifying this rule.

## 3.11 Singular Points

A point at which is **not analytic** is called a **singular point**.

There are several types of singular points.

### Isolated Singularity

The point at which is not analytic is called an **isolated singular point** if we can define a **neighbourhood** of , , in which there are no singular points. Otherwise, is a **non-isolated singular point**.

### Poles

If we can find a **positive integer**, , such that

and is analytic at , then is called a **pole of order** .

If , is called a **simple pole**. Essentially, if the limiting value at a singular point is not , that point is a pole.

For example,

The above equation is clearly not analytic at .

Thus, the singularity at is a pole of order .

Similarly, we have a pole of order at and simple poles at and .

### Branch Points

If is a **multiple-valued function** at , which is a point at which is not analytic, then is a **branch point**.

For example, in , is a branch point. In , and are branch points.

### Removable Singularities

A singular point, , is said to be a **removable singularity** of if exists.

For example, the singular point of is a removable singularity, since .

### Essential Singularities

A singularity which is **not a pole, branch point or a removable singularity** is called an **essential singularity**.

For example, has an essential singularity at .

### Singularities at Infinity

If , then we say has singularities at . This type of singularity is the same as that of at .

For example, the function has a pole of order at , since has a pole of order at .